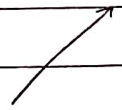
VECTOR

29/08/2022

Defn: A physical pty. having both dirxn & mag. that obeys vector law of addn.

- Representation - 

- Equality of vectors - Equal mag. & same dirxn

- Zero vector ($\vec{0}$) - Mag. - 0
  Dirxn - Unspecified

- Unit vector ($\hat{u}$) - Mag. - 1

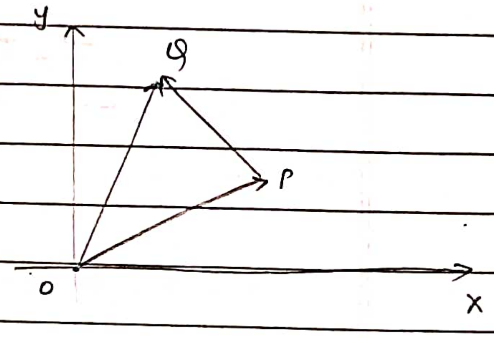
If $\vec{v}$ is a vector s.t its mag. is v & is along $\vec{u}$, then

$$\boxed{\vec{v} = v\,\hat{u}}$$

- Post. vector - Shows post. of a pt. wrt a specified coordinate system.
  It is a fixed vector, i.e its tail is fixed to origion

- Free vector - Translation independent vector.

NOTE: All free vectors can be expressed in terms of post. vectors

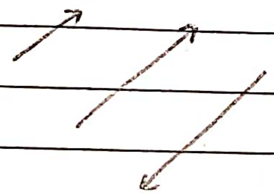


$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

$\Rightarrow \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

i.e $\quad \overrightarrow{PQ}$ = Post. vector(Q) − Post. vector(P)

- <u>Collinearity of Vectors</u> − Parallel or Antiparallel vectors



If 2 vectors $\vec{v}$ & $\vec{u}$ are collinear,
then $\exists\ k \in \mathbb{R}$ s.t

$$\boxed{\vec{v} = k\vec{u}}$$

OPERATIONS

→ **Scalar Product (Dot Product)**

$$\boxed{\vec{v} \cdot \vec{u} = v u \, c_\theta}$$

(Angle enclosed by $\vec{u}$ & $\vec{v}$)

$\theta \in [0, \pi]$

• **Pts :-**

If $\vec{a} = \langle a_1 \;\; a_2 \;\; a_3 \rangle$ & $\vec{b} = \langle b_1 \;\; b_2 \;\; b_3 \rangle$
then

1. $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

2. $c_\theta = \dfrac{\vec{a} \cdot \vec{b}}{ab}$ $\Rightarrow$ Acute angle $\Leftrightarrow$ $\vec{a} \cdot \vec{b} > 0$
Obtuse angle $\Leftrightarrow$ $\vec{a} \cdot \vec{b} < 0$

3. $a^2 = \vec{a} \cdot \vec{a}$

$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})(\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$
$= a^2 + 2\vec{a} \cdot \vec{b} + b^2$

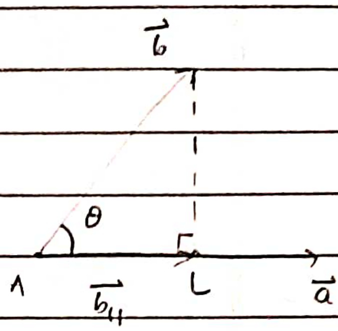
4. Cauchy - Schwartz Inequality

$$(\vec{a} \cdot \vec{b})^2 \leq a^2 b^2$$

$\Rightarrow (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$

5. Projection of $\vec{b}$ on $\vec{a}$

$$b_{\parallel} = b\cos\theta = \frac{ab\cos\theta}{a} = \frac{\vec{a}\cdot\vec{b}}{a} = \vec{b}\cdot\hat{a}$$



$$\vec{b_{\parallel}} = b_{\parallel}\hat{a} = \frac{\vec{a}\cdot\vec{b}}{a}\hat{a}$$

$$= \frac{\vec{a}\cdot\vec{b}}{a^2}\vec{a}$$

→ Vector Product (Cross Product)

$$\boxed{\vec{u}\times\vec{v} = (uv\sin\theta)\,\hat{n}}$$

(Normal vector to the plane made by $\vec{u}$ & $\vec{v}$)

(Angle enclosed by $\vec{u}$ & $\vec{v}$)

$\theta \in [0, \pi]$

If $\vec{w} = \vec{u}\times\vec{v} \Rightarrow \vec{u}, \vec{v}, \vec{w}$ form a right-handed system

- Pts :-

1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

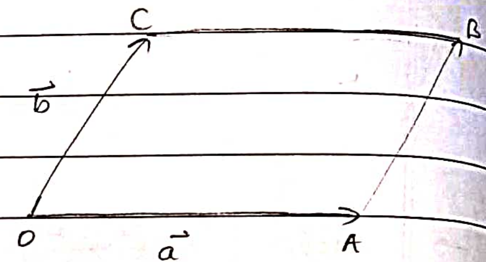
2. $\vec{a}\times\vec{a} = 0$ $\quad (\because \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0)$

3. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

4. $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. $\vec{a} \times \vec{b} = 0 \iff \vec{a}$ & $\vec{b}$ are collinear
(given they are non-zero vectors)

6. $$ar(▱OABC) = 2\, ar(\Delta AOC)$$
$$= |\vec{a} \times \vec{b}|.$$
$$= ab\, s\theta$$



7. Unit vector normal to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

A vector of mag. λ normal to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \lambda \left(\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \right)$

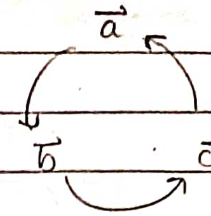
→ Scalar Triple Product (Box Product)

$$(\vec{u} \times \vec{v}) \cdot \vec{w}$$

denoted by $[\vec{a} \ \vec{b} \ \vec{c}]$

• Pts –

1. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$



2. $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

3. $[k_1\vec{a}\ \ k_2\vec{b}\ \ k_3\vec{c}] = k_1k_2k_3\,[\vec{a}\ \vec{b}\ \vec{c}]$

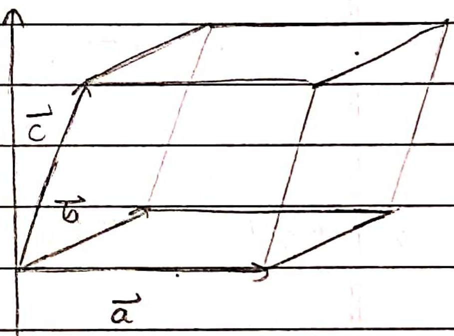
4. $[\vec{a}\ \vec{a}\ \vec{a}] = 0$

5. $[\vec{a}\ \vec{a}\ \vec{b}] = 0$

6. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{c}\ \vec{b}\ \vec{a}]$

7. Geometrical significance

Vol. of parallelapiped formed by edges $\vec{a}, \vec{b}, \vec{c}$ is $|[\vec{a}\ \vec{b}\ \vec{c}]|$



8. The necessary & sufficient condn for 3 non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

→ Scalar product of 4 vectors

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

$$[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} \vec{a}\cdot\vec{u} & \vec{b}\cdot\vec{u} & \vec{c}\cdot\vec{u} \\ \vec{a}\cdot\vec{v} & \vec{b}\cdot\vec{v} & \vec{c}\cdot\vec{v} \\ \vec{a}\cdot\vec{w} & \vec{b}\cdot\vec{w} & \vec{c}\cdot\vec{w} \end{vmatrix}$$

$\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar if

$$[\vec{a}\ \vec{d}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{d}] + [\vec{d}\ \vec{a}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

→ Vector Triple Product

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$$

→ Vector Product of 4 vectors

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\,\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{d}$$

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COLLINEARITY & COPLANARITY

- **Cond^n of Collinearity :**

For 2 vectors — $\vec{a} = \lambda \vec{b}$

For 3 points — $\lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 \vec{c} = 0$ (section formula)

(posit. vectors) where $\lambda_1 + \lambda_2 + \lambda_3 = 0$

$[A(\vec{a}), B(\vec{b}), C(\vec{c})]$

- **Fundamental Theorem of Planarity** — ($\vec{a}, \vec{b} \rightarrow$ non-zero)

If $\vec{a}$ & $\vec{b}$ are 2 non-collinear coplanar vectors, any vector $\vec{c}$ coplanar with $\vec{a}$ & $\vec{b}$ can be uniquely expressed as a linear combination of $\vec{a}$ & $\vec{b}$

i.e $\exists$ unique $\lambda_1, \lambda_2 \in \mathbb{R}$, s.t

$$\vec{c} = \lambda_1 \vec{a} + \lambda_2 \vec{b}$$

For 4 points — $\lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 \vec{c} + \lambda_4 \vec{d} = 0$

(posit. vectors) where $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$

$[A(\vec{a}), B(\vec{b}), C(\vec{c}), D(\vec{d})]$

- **Fundamental Theorem in Space** — ($\vec{a}, \vec{b}, \vec{c} \rightarrow$ non-zero)

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then any vector $\vec{r}$ can be uniquely expressed as linear combination of $\vec{a}, \vec{b}, \vec{c}$

i.e $\exists$ unique $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, s.t

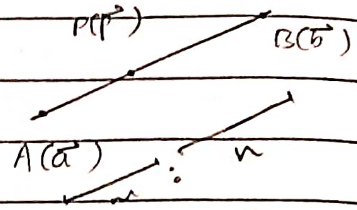
$$\vec{r} = \lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 \vec{c}$$

06/09/2023

- **Section Formula** –

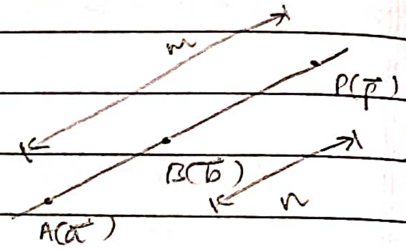
Internal division

$$\vec{p} = \frac{n\vec{a} + m\vec{b}}{n+m}$$



External division

$$\vec{p} = \frac{n\vec{a} - m\vec{b}}{n-m}$$



Q. Find a vector $\vec{c}$ on the plane of $\vec{a} = \langle 2 \;\; 1 \;\; -1 \rangle$ & $\vec{b} = \langle -1 \;\; 1 \;\; 1 \rangle$ s.t $\vec{c}$ is $\perp$ to $\vec{b}$ & $\vec{c} \cdot \langle -2 \;\; 3 \;\; -1 \rangle = -1$

A $\vec{c} = A\vec{a} + B\vec{b} \Rightarrow \vec{b} \cdot \vec{c} = A\,\vec{a} \cdot \vec{b} + B\,\vec{b} \cdot \vec{b}$

$\Rightarrow -2A + 3B = 0$

$\vec{c} \langle -2 \;\; 3 \;\; -1 \rangle = A \langle 2 \;\; 1 \;\; -1 \rangle \langle -2 \;\; 3 \;\; -1 \rangle$
$+ B \langle -1 \;\; 1 \;\; 1 \rangle \langle -2 \;\; 3 \;\; -1 \rangle$

$\Rightarrow 4B = -1 \Rightarrow B = -\frac{1}{4} \Rightarrow A = -\frac{3}{8}$

$\vec{c} = -\frac{3}{8} \langle 2 \;\; 1 \;\; -1 \rangle - \frac{2}{8} \langle -1 \;\; 1 \;\; 1 \rangle$

$= \frac{1}{8} \langle -4 \;\; -5 \;\; 1 \rangle$

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- **Linear Independent system of vectors**

A set of vectors $\vec{a}_i$ $(i = 1, 2, \dots, n)$ is said to be linearly indep. sys. of vectors if $\exists$ $\lambda_i \in \mathbb{R}$ s.t

$$\boxed{\sum \lambda_i \vec{a}_i = 0}$$
$$\Updownarrow$$
$$\boxed{\lambda_i = 0} \quad \forall\ i$$

- **Linear Dep. sys. of vectors**

A set of vectors $\vec{a_i}$ $(i = 1, 2, \dots, n)$ is said to be linear dep. sys. of vectors if $\exists$ λ_i not all zero s.t

$$\boxed{\sum \lambda_i \vec{a_i} = 0}$$

Q. (i) $\vec{a} = \langle 1 \quad 1 \quad 1 \rangle$, $\vec{b} = \langle 4 \quad 3 \quad 4 \rangle$
$\vec{c} = \langle 1 \quad x_1 \quad x_2 \rangle$
Find x_1, x_2 s.t $\vec{a}, \vec{b}, \vec{c}$ are linearly dep
& $|\vec{c}| = \sqrt{3}$

(ii) $\langle 1 \quad 1 \quad 1 \rangle$, $\langle 2 \quad 3 \quad -1 \rangle$, $\langle -1 \quad -2 \quad 2 \rangle$ are linearly dep. Prove it.

A. (i) $1 + x_1^2 + x_2^2 = 3 \Rightarrow x_1^2 + x_2^2 = 2$

$\exists\ \lambda_i$ s.t $\sum \lambda_i \vec{a_i} = 0 \Rightarrow \langle \lambda_1 + 4\lambda_2 + \lambda_3 \quad \lambda_1 + 3\lambda_2 + x_1\lambda_3 \quad \lambda_1 + 4\lambda_2 + x_2\lambda_3 \rangle$
$= \langle 0 \quad 0 \quad 0 \rangle$

$\Rightarrow \underline{x_2 = 1}$

$x_1^2 + x_2^2 = 2 \Rightarrow x_1^2 = 1 \Rightarrow x_1 = \pm 1$

(ii) $\langle 2 \quad 3 \quad -1 \rangle = \langle 1 \quad 1 \quad 1 \rangle - \langle -1 \quad -2 \quad 2 \rangle$

$\Rightarrow \langle 1 \quad 1 \quad 1 \rangle + (-1) \langle -1 \quad -2 \quad 2 \rangle$
$+ (-1) \langle 2 \quad 3 \quad -1 \rangle = 0$

Q (i) $\vec{a}$ & $\vec{b}$ are mutually $\perp$ unit vectors & $\vec{r}$ is any vector satisfying $\vec{r} \cdot \vec{a} = 0$ & $\vec{r} \cdot \vec{b} = 1$ & $[\vec{r} \; \vec{a} \; \vec{b}] = 1$
Find $\vec{r}$

(ii) If $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ & $\vec{r} \cdot \vec{c} = 0$ provided $\vec{c} \cdot \vec{b} \neq 0$, find $\vec{r}$

(iii) Let $\hat{a}, \hat{b}, \hat{c}$ be unit vectors s.t. $\hat{a} \times \hat{b} = \hat{c}$
$[\vec{r} \; \hat{b} \; \hat{c}] = 3$, $[\vec{r} \; \hat{c} \; \hat{a}] = 4$ & $[\vec{r} \; \hat{a} \; \hat{b}] = 2$
Find $\vec{r}$.

A (i) ★ $\vec{r} = \lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 (\vec{a} \times \vec{b})$

$\vec{r} \cdot \vec{b} = \lambda_1 \cancel{\vec{a} \cdot \vec{b}}^{(0)} + \lambda_2 b^2 = 1 \Rightarrow \underline{\lambda_2 = 1}$

$\vec{r} \cdot \vec{a} = \lambda_1 a^2 + \cancel{\lambda_2 \vec{a} \cdot \vec{b}}\,(0) \Rightarrow \underline{\lambda_1 = 0}$

$[\vec{r} \;\; \vec{a} \;\; \vec{b}] = \vec{r} \cdot (\vec{a} \times \vec{b}) = 1$

$\Rightarrow (\vec{b} + \lambda_3 (\vec{a} \times \vec{b})) \cdot (\vec{a} \times \vec{b}) = 1$

$\Rightarrow \lambda_3 \underbrace{|(\vec{a} \times \vec{b})|}_{1}{}^2 = 1 \Rightarrow \underline{\lambda_3 = 1}$

$$\underline{\vec{r} = \vec{b} + (\vec{a} \times \vec{b})}$$

(ii) $(\vec{r} - \vec{a}) \times \vec{b} = 0 \Rightarrow \vec{r} - \vec{a} = k\vec{b}$

$\Rightarrow \vec{r} = \vec{a} + k\vec{b}$

$\vec{r} \cdot \vec{c} = \vec{a} \cdot \vec{c} + k\, \vec{b} \cdot \vec{c} = 0 \Rightarrow k = -\left(\dfrac{\vec{a} \cdot \vec{c}}{\vec{b} \cdot \vec{c}}\right)$

$$\Rightarrow \underline{\vec{r} = \vec{a} - \left(\dfrac{\vec{a} \cdot \vec{c}}{\vec{b} \cdot \vec{c}}\right) \vec{b}}$$

(iii) $\vec{r} \cdot (\hat{a} \times \hat{b}) = 2 \Rightarrow \vec{r} \cdot \hat{c} = 2$

$ab \sin\theta = c$

$\Rightarrow \underline{\sin\theta = 1}$

$\Downarrow$

$\underline{\hat{a} \cdot \hat{b} = 0}$

$(\hat{a} \times \hat{b}) \times \hat{a} = a^2 \hat{b} - (\hat{a} \cdot \hat{b})\hat{a} = \hat{c} \times \hat{a}$

$\Rightarrow \underline{\hat{c} \times \hat{a} = \hat{b}}$

similarly $\hat{b} \cdot \hat{c} = 0$

& $\hat{c} \cdot \hat{a} = 0$

$\vec{r} \cdot (\hat{c} \times \hat{a}) = 4 \Rightarrow \vec{r} \cdot \hat{b} = 4$

similarly $\vec{r} \cdot \hat{a} = 3$

$\Rightarrow \lambda_1 = 3, \; \lambda_2 = 4, \; \lambda_3 = 2$

$\vec{r} = \lambda_1 \hat{a} + \lambda_2 \hat{b} + \lambda_3 \hat{c} \Rightarrow \underline{\vec{r} = 3\hat{a} + 4\hat{b} + 2\hat{c}}$

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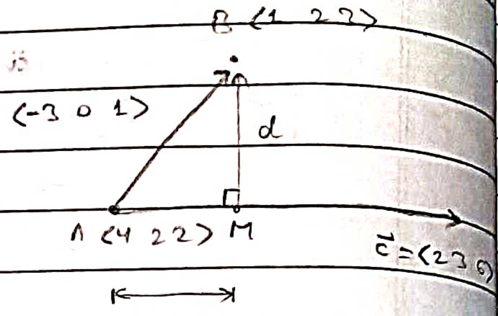
Q. Find the dist. of pt. B<1 2 3> from the line which is passing through A<4 2 2> & which is parallel to the vector $\vec{c} = \langle 2\ 3\ 6\rangle$

A. $d = \sqrt{AB^2 - AM^2}$

$= \underline{\sqrt{10}}$



$\frac{1}{7}\langle -3\ 0\ 1\rangle \langle 2\ 3\ 6\rangle = 0$

Q. $P = \langle 1\ 1\ 1\rangle$

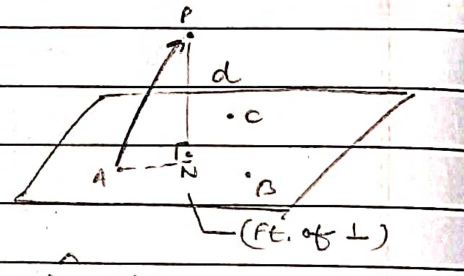
$A = \langle 2\ 1\ 1\rangle$

$B = \langle 1\ 2\ 1\rangle$

$C = \langle 1\ 1\ 2\rangle$

Find dist. of plane of ABC from P. & the foot of $\perp$

A. $\overline{PA} = \langle 1\ 0\ 0\rangle$



$d = \overline{PA} \cdot \hat{NP}$

$= \langle 1\ 0\ 0\rangle \cdot \frac{1}{\sqrt{3}}\langle 1\ 1\ 1\rangle$

$= \left(\frac{1}{\sqrt{3}}\right)$

$\hat{NP} = (\widehat{\overline{AB} \times \overline{BC}}) = \frac{1}{\sqrt{3}}\langle 1\ 1\ 1\rangle$

① N on normal

$\overline{PN} = k\hat{n} \Rightarrow \overline{ON} = (1-k)\langle 1\ 1\ 1\rangle$

② N in plane

$[\overline{AB}\ \ \overline{BC}\ \ \overline{PN}] = 0$

$\Rightarrow \quad 2 \quad 1 \quad 1$

Q. $\vec{b}$ & $\vec{c}$ are non-collinear vectors. If $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a}\cdot\vec{b})\vec{b}$
$= (4 - 2x + sy)\vec{b} + (x^2-1)\vec{c}$

& $(\vec{c}\cdot\vec{c})\vec{a} = \vec{c}$.

Find ordered pairs (x, y)

A. $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{b} = (4 - 2x + sy)\vec{b} + (x^2-1)\vec{c}$

$\Rightarrow \quad (\vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{b} - 4 + 2x - sy)\vec{b} + (1 - \vec{a}\cdot\vec{b} - x^2)\vec{c} = 0$

$\because$ $\vec{b}$ & $\vec{c}$ are non-collinear $\Rightarrow$ $\vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{b} - 4 + 2x - sy = 0$

& $1 - \vec{a}\cdot\vec{b} - x^2 = 0$

$\Rightarrow \quad sy - 2x + 4 = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 1 + \vec{a}\cdot\vec{b}$

$x^2 - 1 = -\vec{a}\cdot\vec{b}$

$c^2\vec{a} = \vec{c} \quad \Rightarrow \quad c^2(\vec{a}\cdot\vec{c}) = c^2 \Rightarrow \underline{\vec{a}\cdot\vec{c} = 1}$

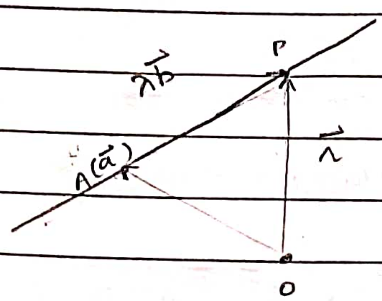
$sy - 2x + 4 = 1 + 1 - x^2 \quad \Rightarrow \quad sy = -1 - (x-1)^2$

$\underline{x = 1}$ & $\underline{y = \frac{(4n+1)\pi}{2}}$

FORMULAE

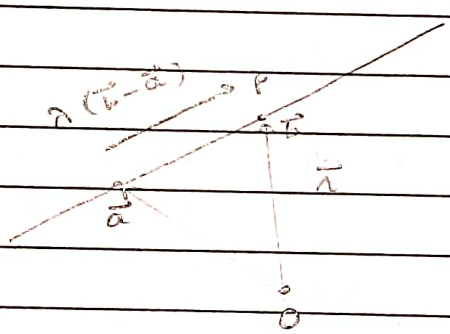
1. Eqn of line passing through $A(\vec{a})$ & parallel to $\vec{b}$ is given by

$$\boxed{\vec{r} = \vec{a} + \lambda\vec{b}} \quad , \quad \lambda \in \mathbb{R}$$



2. Line passing through $\vec{a}$ & $\vec{b}$

$$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})} \quad , \quad \lambda \in \mathbb{R}$$



3. $L_1 : \vec{r} = \vec{a} + \mu_1\vec{b}$ & $L_2 : \vec{r} = \vec{a} + \mu_2\vec{b}$ intersecting at $A(\vec{a})$, the angle bisector is given by

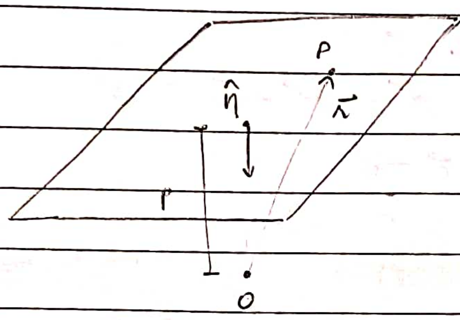
$$\boxed{\vec{r} = \vec{a} + \lambda\left(\frac{\vec{b}}{b} \pm \frac{\vec{c}}{c}\right)} \quad , \quad \lambda \in \mathbb{R}$$

4. Eqn of plane (Normal form)

$$\boxed{\vec{r} \cdot \hat{n} = p}$$

(unit vector along normal to plane) — $\hat{n}$

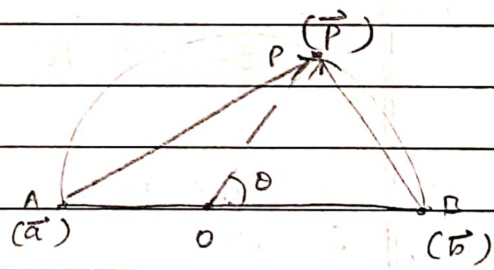
p — (length of $\perp$ drawn from origin to the plane)



5. Plane passing through A ($\vec{a}$)

$$\boxed{(\vec{r} - \vec{a}) \cdot \hat{n} = 0}$$

Q. Prove that $\angle$ in a semi-circle is 90°.



A. $|\vec{p}| = |\vec{a}| = |\vec{b}| = r$

$\overline{PB} \cdot \overline{PA} = (\vec{p} - \vec{a}) \cdot (\vec{p} - \vec{b})$

$= |\vec{p}|^2 - \vec{a}\cdot\vec{p} - \vec{b}\cdot\vec{p} + \vec{a}\cdot\vec{b}$

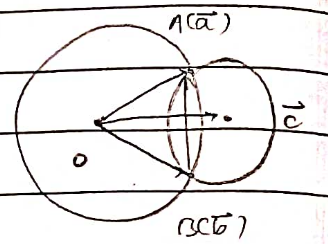
$= r^2 - \underbrace{(\vec{a}+\vec{b})}_{0}\vec{p} + r^2 \cos\pi$

$= 0$

$\Downarrow$

PB $\perp$ PA

Q. Prove that common chord of 2 intersecting circles is $\perp$ to the line joining their centres.



A. <u>To prove</u> : $\vec{c}\cdot(\vec{a}-\vec{b})=0$

given $|\vec{a}-\vec{c}|=|\vec{b}-\vec{c}|$

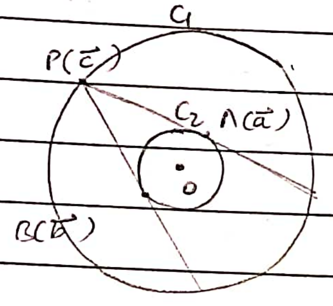
& $|\vec{a}|=|\vec{b}|$

$(\vec{a}-\vec{c})^2=(\vec{b}-\vec{c})^2$

$\Rightarrow \cancel{a^2}-2\vec{a}\cdot\vec{c}+\cancel{c^2}=\cancel{b^2}-2\vec{b}\cdot\vec{c}+\cancel{c^2}$

$\Rightarrow (\vec{a}-\vec{b})\cdot\vec{c}=0$

Q.



Prove that centroid of ΔAPB lies on C_2

A. $\vec{G}=\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

To prove : $|\vec{G}|=|\vec{a}|=|\vec{b}|=r$

given $|\vec{c}|=R$

$|\vec{c}-\vec{a}|=|\vec{c}-\vec{b}|$

$9G^2=(\vec{a}+\vec{b}+\vec{c})^2=a^2+b^2+c^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$

07/09/2021

→ **Reciprocal system of Vectors** –

Let $\vec{a}, \vec{b}, \vec{c}$ be non-coplanar system of vectors. The system of vectors $\vec{a}', \vec{b}', \vec{c}'$ s.t

$$\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$$

&

$$\vec{a}\cdot\vec{b}' = \vec{a}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0$$

is called its reciprocal system of vectors.

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} \qquad \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \qquad \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

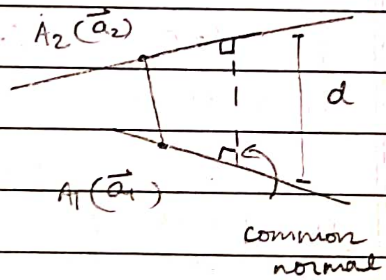
NOTE: 1. $\vec{a}', \vec{b}', \vec{c}'$ form a non-coplanar system of vectors

2. $\vec{r} = \lambda_1\vec{a}' + \lambda_2\vec{b}' + \lambda_3\vec{c}'$

• **Shortest distance b/w 2 Skew (Non-coplanar) lines**

$L_1: \vec{r_1} = \vec{a_1} + \lambda_1\vec{b_1}$

$L_2: \vec{r_2} = \vec{a_2} + \lambda_2\vec{b_2}$

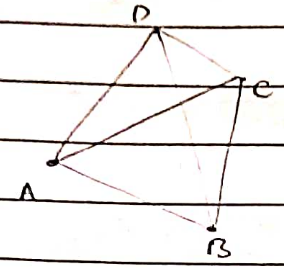


$$d = \left(\text{Projection of } A_1A_2 \text{ on Common Normal}\right)$$

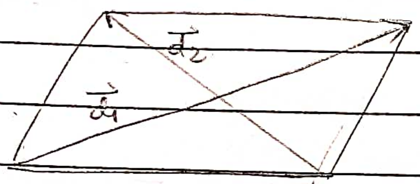
$$= \boxed{(\vec{a_2}-\vec{a_1})\cdot\left(\frac{\vec{b_1}\times\vec{b_2}}{|\vec{b_1}\times\vec{b_2}|}\right)}$$

→ Results

1. Vol. of Tetrahedron $= \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

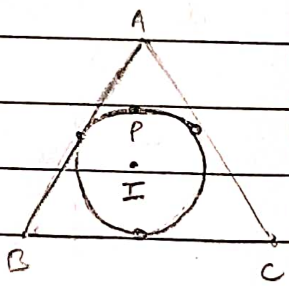


2. Area of Quadrilateral with diagonals $\vec{d_1}$ & $\vec{d_2}$ $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$



Q. Consider an eq. ΔABC, side $= l$
If P be any pt. on the incircle of ΔABC, then prove using vectors that $PA^2 + PB^2 + PC^2 = k$ & $k = \left(\frac{5l^2}{4}\right)$

A. To prove : $\vec{PA}^2 + \vec{PB}^2 + \vec{PC}^2 = k$



$\vec{PA} = \vec{IA} - \vec{IP}$
$\vec{PB} = \vec{IB} - \vec{IP}$
$\vec{PC} = \vec{IC} - \vec{IP}$

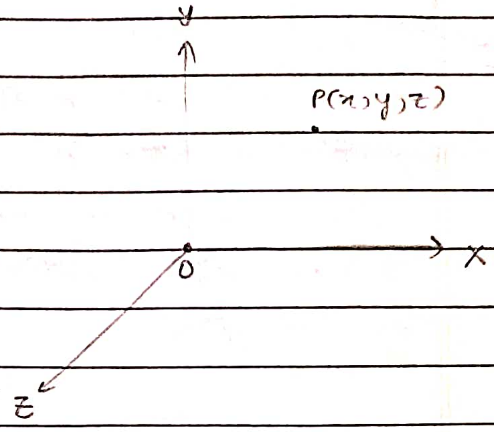
$\Sigma \vec{PA}^2 = \Sigma \vec{IA}^2 + 3\vec{IP}^2 - 2\vec{IP}\overbrace{(\Sigma \vec{IA})}^{0}$ (∵ in eq Δ I=O=G)

$= 3r^2 + \frac{3l^2}{4} + 3r^2 = \left(\frac{5l^2}{4}\right)$

3D GEOMETRY

08/09/2023

- Pts :-



1. Dist. formula

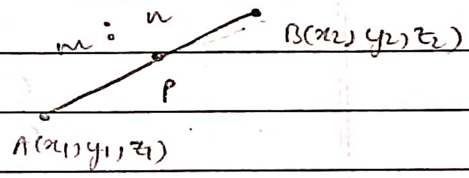
$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

(octane system of coordinates)

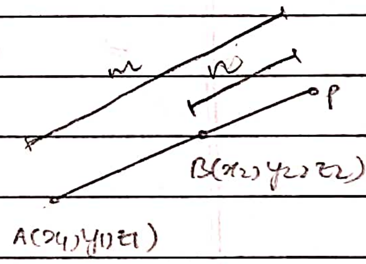
2. Section formula

2.1 Internal division

$$P \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n} \right)$$



2.2 External division



$$P \equiv \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n} \right)$$

3. Centroid of Δ

$$G \equiv \left(\frac{\Sigma x_i}{3}, \frac{\Sigma y_i}{3}, \frac{\Sigma z_i}{3} \right)$$

4. Direction Cosines

If α, β, γ be the angles made by a line with the x, y, z axes respectively, then $C\alpha, C\beta, C\gamma$ are called dirn cosines.

$l = C\alpha$

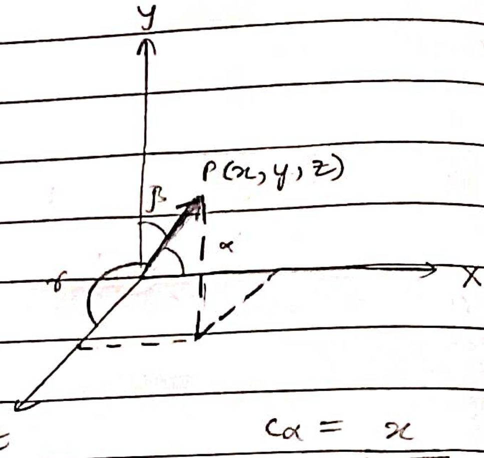
$m = C\beta$

$n = C\gamma$

$$\boxed{l^2+m^2+n^2=1}$$

5. **Direction Ratio**

Any 3 nos. a, b, c prop. to the dir^n cosines l, m, n are called dir^n ratios of a line



$$\boxed{\frac{l}{a} = \frac{m}{b} = \frac{n}{c}}$$

$$C\alpha = \frac{x}{\sqrt{\Sigma x^2}}$$

$$C\beta = \frac{y}{\sqrt{\Sigma x^2}}$$

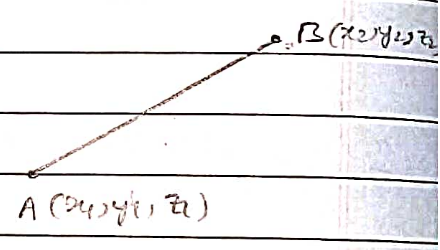
$$C\gamma = \frac{z}{\sqrt{\Sigma x^2}}$$

6. Rel^n b/w D.C & D.R

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \quad l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

7. D.C of a line joining 2 pts.



(Depending on whether choosing AB or AB)

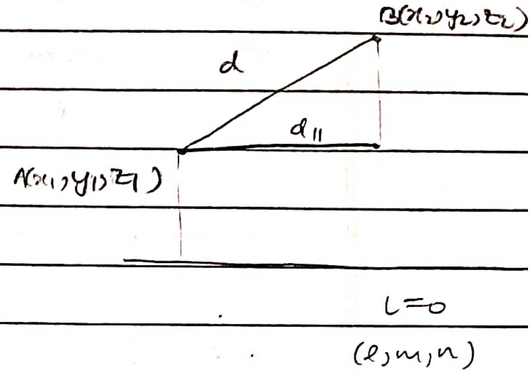
$$l = \frac{\pm(x_2 - x_1)}{\sqrt{\Sigma(x_2-x_1)^2}}$$

$$m = \frac{\pm(y_2 - y_1)}{\sqrt{\Sigma(x_2-x_1)^2}}$$

$$n = \frac{\pm(z_2 - z_1)}{\sqrt{\Sigma(x_2-x_1)^2}}$$

8. Projection of a line segment on another line

$$d_{11} = \sqrt{l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)}$$

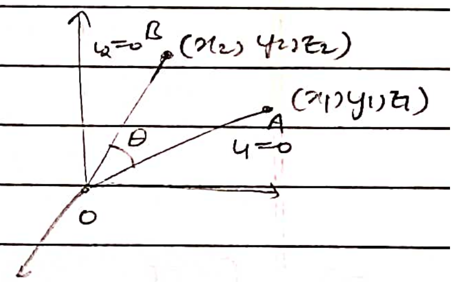


9. Angle b/w 2 lines

$$C\theta = \frac{OA^2 + OB^2 - AB^2}{2(OA)(OB)}$$

$$= \frac{\Sigma x_1^2 + \Sigma x_2^2 - \Sigma(x_1-x_2)^2}{2\left(\sqrt{\Sigma x_1^2}\right)\left(\sqrt{\Sigma x_2^2}\right)}$$

$$= \frac{\Sigma x_1 x_2}{\sqrt{\Sigma x_1^2}\cdot\sqrt{\Sigma x_2^2}} = \Sigma \frac{x_1}{\sqrt{\Sigma x_1^2}} \frac{x_2}{\sqrt{\Sigma x_2^2}}$$



$$C\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$C\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\left(\sqrt{\Sigma a_1^2}\right)\left(\sqrt{\Sigma a_2^2}\right)}$$

$\Sigma l_1 l_2 = 0 \iff$ lines $\perp$

Date
Page

→ **Straight line**

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = \lambda$$

$l, m, n \rightarrow$ DC/DRs of the line

NOTE: Any pt. on the line can be written as $(x_1+\lambda l, y_1+\lambda m, z_1+\lambda n)$

→ **Plane**

Def: A plane is a surface s.t if any 2 pts. are taken on it, the line segment joining them lies completely on the surface

$$ax+by+cz+d=0$$

$a, b, c \rightarrow$ DC/DRs to the normal of plane

Clearly, at least one of a, b, c must not be zero

$x=0 \rightarrow$ yz plane
$y=0 \rightarrow$ zx plane
$z=0 \rightarrow$ xy plane

$x=k \rightarrow$ ∥ to yz plane
$y=k \rightarrow$ ∥ to zx plane
$z=k \rightarrow$ ∥ to xy plane

Plane $\parallel$ to :

x-axis	-	$by + cz + d = 0$
y-axis	-	$ax + cz + d = 0$
z-axis	-	$ax + by + d = 0$

- Intercept form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Intercepts to :

x-axis	-	$(a, 0, 0)$
y-axis	-	$(0, b, 0)$
z-axis	-	$(0, 0, c)$

- Normal form

$$lx + my + nz = p$$

$l, m, n \rightarrow$ DCs of normal to plane

$p \rightarrow$ Dist. of origin from plane

$ax + by + cz + d = 0 \Rightarrow \frac{a}{\sqrt{\Sigma a^2}} x + \frac{by}{\sqrt{\Sigma a^2}} + \frac{cz}{\sqrt{\Sigma a^2}} = \frac{-d}{\sqrt{\Sigma a^2}}$

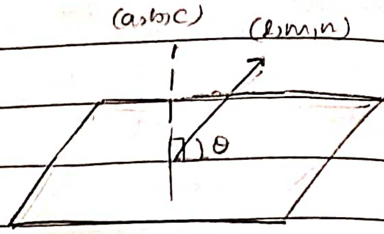
(Normal Form) [RHS > 0]

- Angle b/w line & plane

$$s\theta = \frac{al+bm+cn}{(\sqrt{\Sigma a^2})(\sqrt{\Sigma n^2})}$$

$\downarrow$

$c(90^\circ - \theta)$



line $\parallel$ to plane $\Leftrightarrow$ $\Sigma al = 0$

line $\perp$ to plane $\Leftrightarrow$ $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$

- Angle b/w 2 planes

($\angle$ b/w planes) = ($\angle$ b/w normals)

P_1: $a_1x + b_1y + c_1z + d_1 = 0$

P_2: $a_2x + b_2y + c_2z + d_2 = 0$

$$c\theta = \frac{\Sigma a_1a_2}{(\sqrt{\Sigma a_1^2})(\sqrt{\Sigma a_2^2})}$$

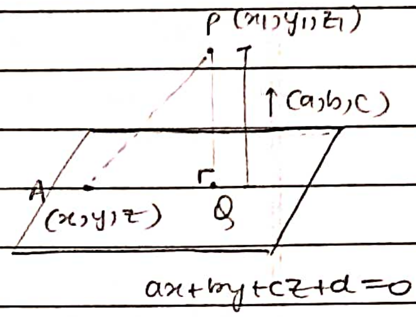
Plane $\parallel$ to plane : $ax + by + cz + \lambda_1 = 0$

$ax + by + cz + \lambda_2 = 0$

- Family of Planes – Set of planes passing through a common line

- $\perp$ dist. of pt. from plane

$$d = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}} \right|$$



Proof: d = Projection of AP on QP

$$= \left| \frac{a(x_1-x)}{\sqrt{\Sigma a^2}} + \frac{b(y-y_1)}{\sqrt{\Sigma a^2}} + \frac{c(z-z_1)}{\sqrt{\Sigma a^2}} \right|$$

$$= \left| \frac{ax_1+by_1+cz_1}{\sqrt{\Sigma a^2}} - \frac{ax+by+c}{\sqrt{\Sigma a^2}} \right|$$

$$= \left| \frac{ax_1+by_1+cz_1+d}{\sqrt{\Sigma a^2}} \right|$$

- Dist b/w 2 $\parallel$ planes

$P_1: ax+by+cz+d_1=0$

$P_2: ax+by+cz+d_2=0$ $\Rightarrow$ $\left| \frac{d_2-d_1}{\sqrt{a^2+b^2+c^2}} \right|$

- Angle Bisector

$$\left(\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \right) = \pm \left(\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \right)$$